JOURNAL OF COMPUTATIONAL PHYSICS 76, 281–300 (1988)

Random Vortex Methods for the Navier–Stokes Equations

CHIEN-CHENG CHANG

Lawrence-Berkeley Laboratory and Department of Mathematics, University of California, Berkeley, California 94720

Received December 24, 1986; revised May 19, 1987

Two random vortex methods of Runge–Kutta type are presented for solving the two-dimensional Navier–Stokes equations. We investigate the accuracy of these methods by considering the model problem of a rotating flow with intitial vorticity concentrated uniformly on a disk of finite radius. Functionals of the numerical solution are computed by Monte Carlo estimates with efficient variance reduction, and the results are compared to those obtained from Euler's method. The numerical results show that both of the methods produce errors smaller by one power of the time step size than Euler's method, one seemingly even better than the other. These Runge–Kutta methods are derivations of similar schemes proposed by us in an earlier time for solving stochastic differential equations with constant diffusion coefficients. © 1988 Academic Press, Inc

1. INTRODUCTION

In this article we present three random vortex methods for the two-dimensional Navier–Stokes equations and compare their accuracy in numerical examples. The methods that we consider are Euler's method and two other methods—A and B, say—based on the midpoint rule. Methods A and B are derived from similar schemes—called A_0 and B_0 here—proposed by us in an earlier time for solving stochastic differential equations (SDEs) [9, 10]. Method B_0 has order 1.5 in the L_2 sense while Method A_0 was conjectured to have order 2 in a weak sense. Roughly speaking, the derivations are due to SDEs' capability of modeling the physics of convection by a drift term and of viscous diffusion by a Wiener process (Brownian motion).

Our model problem is a two-dimensional rotating flow with initial vorticity distributed uniformly on a disk of finite radius $a(=0.5)$. The viscosity $\nu(=0.002)$ is chosen within the range of a typical slightly viscous flow. Following [31, 33], we compare the methods by estimating two functionals of the flow field which can be evaluated exactly. Indeed, as we shall see in Section 2, the flow field is actually a functional of the Wiener process. The total time interval $T(=4)$ of the test is about the order of the period of rotation of the corresponding inviscid flow.

There are two main sources of error in the stochastic schemes for estimating the functionals: the time discretization and the random sampling. The former produces local truncation error at each time step which depends only on the specific scheme

0021-9991/88 $3.00

used and can be reduced to some extent by decreasing the time step size. The latter comes from Monte Carlo estimation of the functionals. The error variance of a typical Monte Carlo estimate is usually proportional to $1/N$ so that the sampling error often dominates the discretization error, and therefore should be minimized as much as possible. A variance reduction technique is derived here; we make successive time differences of the functionals and employ the non-anticipating property of SDEs.

Other factors affecting the calculations include the cutoff δ, the cutoff function f_δ and the number N ($= 856$) of the vortex blobs used. In order to exhibit the accuracy of the schemes, the cutoffs are chosen so that their effects are minimal, and, instead of being randomly distributed, the initial vortex blobs are placed on a uniform grid, as suggested by Roberts [33].

Our numerical results show that the sampling errors for the usual Monte Carlo estimate in Euler's method are roughly of the same order as the time discretization errors but dominate in the other two. As expected, this estimate cannot show the high accuracy of Methods A and B for such relatively small sample ($N=856$) without efficient variance reduction. The variance reduction technique mentioned above produces for Method A errors of the same order as those by a usual Monte Carlo estimate and exhibits clearly the first-order accuracy of the method. However, this technique reduces drastically errors for both Methods A and B. Numerical evidence shows that either Method A or B is, practically a second-order scheme for the Navier–Stokes equations as long as an efficient variance reduction technique is available.

The first random vortex method was conceived by Chorin and used to study a slightly viscous flow with boundary conditions [13]. This method consists of solving Euler's equations by a vortex method and sampling Gaussian random variables to model the diffusion equation. It is therefore a fractional step type method. The legitimacy of using methods of this kind has been studied by several authors, e.g., [11, 17, 36]. A vortex method for solving Euler's equations can be briefly described as follows. In a vortex method, the initial vorticity field is partioned to a sum of vortex blobs called vortices; and Euler's equations are replaced by a finite set of ordinary differential equations according to which the vortices evolve. Therefore both the vortex method and the random vortex method are grid free, i.e., no spatial discretization is needed to advance the vortex blobs.

The main difference between Chorin's method and our Methods A and B lies in that at each time step the velocity field in the latter case is set to be interlaced with the purely random field (the Wiener process); that is, intermediate random interactions are introduced. These intermediate effects complicate the numerical diffusion process, requiring our schemes to use more information about the Wiener process. This might be crucial to the success of designing high accuracy random vortex methods. A theoretical study along this direction is still lacking at present.

The random vortex method has been successful in the study of several physical phenomena, for example, turbulent combustion [20, 35]. However, it should be noted that the very physical vortices in the vortex method, except initially, do not

serve in a similar way as in the random vortex method. Unlike solving Euler's equations one cannot, for the Navier–Stokes equations, keep track of the paths of physical vortices by solving a system of ordinary differential equations due to the existence of a viscous term. In a random vortex method, each vortex carries a certain weight determined by the initial vorticity field, and their motion generates at each time step a probability distribution; the velocity field in turn is determined via the distribution and the weights through the Biot–Savart law. In short, the random vortex method is a method that provides an approximation to the velocity field through the distribution of random vortices, each of them being specified by a quadruple: its position, its weight, the cutoff, and the cutoff function.

It is appropriate here for us to mention some fundamental aspects of the vortex method and its development. The success of the vortex method consists in the use of vortex blobs, suggested by Chorin [13]. The early study, by Chorin and Bernard [16], of a vortex method without using vortex blobs—called the point vortex method—showed that the method is unstable in predicting rollup of nonuniform vortex sheets. In a vortex method, the velocity is determined by integrating the vorticity against a kernel with singularity at the origin. The above instability is thus due to when two vortex blobs come very close to each other. The idea of using vortex blobs is therefore to cut off this unphysical singularity. For this purpose, a class of so-called cutoff functions was introduced by Hald, which enabled him to give the first convergence proof of vortex method [23, 24]. Subsequently, Beale and Majda [7] designed vortex methods of arbitrary accuracy by careful choice of cutoff functions. For other aspects of the vortex method, especially in three dimensions, we refer to Beale and Majda [5, 6], Leonard [28], Greengard [22], and Anderson and Greengard [2]. Of special interest are Chorin [14, 15], where the idea of the random vortex sheet is introduced, and Anderson [1] who treats flows of slightly variable density.

This paper is organized as follows. In the next section, we derive the random vortex methods in three steps: random equations are introduced, the use of cutoff function is explained, and the fully discretized random vortex algorithms are presented. Then a section is devoted to describing the model problem and the variance reduction technique. Numerical results are then presented and conclusions are drawn. In the Appendix, we briefly describe SDEs and the involved numerical schemes A_0 and B_0.

2. Random Vortex Methods

We begin by showing how a stochastic equation is related to the fluid equations, and give a heuristic discussion on their connection. Consider the two-dimensional Navier–Stokes equations for incompressible fluids with constant density ρ,

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{u}, \tag{2.1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2.2}$$

where $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ is the velocity, $p = p(t, \mathbf{x})$ is the pressure, and the constant ν is the kinematic viscosity. By introducing the vorticity

$$\omega(t, \mathbf{x}) = (\partial_1 u_2 - \partial_2 u_1)(t, \mathbf{x}),$$

where $\mathbf{u} = (u_1, u_2)$, we can rewrite Eqs. (2.1) and (2.2) into a single equation

$$\omega_t + (\mathbf{u} \cdot \nabla)\, \omega = \nu \Delta \omega. \tag{2.3}$$

This equation connects the Navier–Stokes equations with SDEs. Suppose first that the initial vorticity $\omega_0(\mathbf{x}) = \omega(0, \mathbf{x})$ is a probability density on R^2; the general case will be discussed later. Then it follows from (2.2) that (2.3) is the Kolmogorov forward equation for the SDE [3, 21]

$$d\mathbf{Y} = \mathbf{u}(t, \mathbf{Y}(t))\, dt + \sqrt{2\nu}\, d\mathbf{W}_t, \qquad t > 0, \tag{2.4}$$

where $\mathbf{W}_t$ is a normalized Wiener process. The random variable $\mathbf{Y}(t)$ has the distribution

$$P(\mathbf{Y}(t) \in A) = \int_A \omega(t, \mathbf{y})\, d\mathbf{y}, \qquad t \geqslant 0$$

for any Borel set A in R^2. For the sake of completeness, some properties of SDEs and of the Wiener process, and the involved schemes A_0 and B_0 are briefly described in the Appendix.

The connection between (2.3) and (2.4) reveals that the SDE models the convection and diffusion. To take a close look at this, we consider convection and diffusion separately. In the first, we let $\nu = 0$, then (2.3) becomes Euler's equations and (2.4) is equivalent to their characteristic formulation. In the second, we set $\mathbf{u} \equiv 0$ and consider $\omega_0(\mathbf{x}) = \delta(\mathbf{x})$, the Dirac delta function. Then the solution to (2.3) is the diffusion kernel

$$\omega(t, \mathbf{x}) = \frac{1}{4\pi\nu t} \exp\left(-\frac{|\mathbf{x}|^2}{4\nu t} \right).$$

This is exactly the probability density of the solution to (2.4)—$\mathbf{Y}(t) = \sqrt{2\nu}\mathbf{W}_t$. In other words, if initially a vortex is located at the origin, at a later time t it will distribute as the Gaussian random variable $\sqrt{2\nu}\mathbf{W}_t$. Therefore, Eq. (2.4) may be regarded as combining the effect to advance vortices in velocity $\mathbf{u}$ and that to redistribute them in a Gaussian manner. This is the basic idea of random vortex methods.

Next we look for a probabilistic representation of the velocity field. Since the flow is incompressible, we can introduce a stream function $\psi = \psi(t, \mathbf{x})$ such that

$$-\partial_2 \psi(t, \mathbf{x}) = u_1(t, \mathbf{x}), \qquad \partial_1 \psi(t, \mathbf{x}) = u_2(t, \mathbf{x}). \tag{2.5}$$

Then,

$$\omega = \partial_1 u_2 - \partial_2 u_1 = \partial_1^2 \psi + \partial_2^2 \psi = \Delta \psi, \tag{2.6}$$

which is Poisson's equation for ψ. Assume that $\omega(t, \cdot)$ decays rapidly at infinity; the solution of Eq. (2.6) is then given by

$$\psi(t, \mathbf{x}) = (G * \omega)(t, \mathbf{x}) = \frac{1}{2\pi} \int \log(|\mathbf{x} - \mathbf{y}|)\, \omega(t, \mathbf{y})\, d\mathbf{y}, \tag{2.7}$$

where $*$ denotes convolution, and

$$G(\mathbf{x}) = \frac{1}{2\pi} \log(|\mathbf{x}|) \tag{2.8}$$

is the fundamental solution of the Laplace operator. Setting

$$K_1(\mathbf{x}) = (-\partial_2 G)(\mathbf{x}), \qquad K_2(\mathbf{x}) = (\partial_1 G)(\mathbf{x}), \tag{2.9}$$

we have, from (2.7) and (2.9), that

$$\mathbf{u}(t, \mathbf{x}) = (\mathbf{K} * \omega)(t, \mathbf{x}) = \int \mathbf{K}(\mathbf{x} - \mathbf{y})\, \omega(t, \mathbf{y})\, d\mathbf{y}, \tag{2.10}$$

$\mathbf{K} = \mathbf{K}(\mathbf{x})$ being the velocity kernel,

$$\mathbf{K}(\mathbf{x}) = (K_1, K_2)(\mathbf{x}) = \frac{1}{2\pi} \frac{1}{|\mathbf{x}|^2} (-x_2, x_1). \tag{2.11}$$

Equation (2.10) is the Biot–Savart law for the velocity field $\mathbf{u}$. We can further write this equation in Lagrangian form to discover an explicit probabilistic representation for $\mathbf{u}$. Actually, for any flow property, an expression of this kind is crucial for the random vortex method. Define

$$\mathbf{u}(t, \boldsymbol{\sigma}) \equiv \mathbf{u}(t, \mathbf{x}(t, \boldsymbol{\sigma}));$$

then

$$\begin{aligned} \mathbf{u}(t, \boldsymbol{\sigma}) &= \int \mathbf{K}(\mathbf{x}(t, \boldsymbol{\sigma}) - \mathbf{y})\, \omega(t, \mathbf{y})\, d\mathbf{y} \\ &= \int \mathbf{K}(\mathbf{x}(t, \boldsymbol{\sigma}) - \mathbf{y}(t, \boldsymbol{\sigma}'))\, \omega(t, \boldsymbol{\sigma}')\, d\boldsymbol{\sigma}' \\ &= \int \mathbf{K}(\mathbf{x}(t, \boldsymbol{\sigma}) - \mathbf{y}(t, \boldsymbol{\sigma}'))\, dP_{\mathbf{Y}}(\boldsymbol{\sigma}'), \end{aligned} \tag{2.12}$$

or, in Eulerian terms,

$$\mathbf{u}(t, \mathbf{x}) = \int \mathbf{K}(\mathbf{x} - \mathbf{y}(t, \boldsymbol{\sigma}'))\, dP_{\mathbf{Y}}(\boldsymbol{\sigma}'). \tag{2.13}$$

The second equality in (2.12) follows from the incompressibility of the fluid. Here $P_{\mathbf{Y}}$ is the probability measure induced by the random variable $\mathbf{Y} = \mathbf{Y}(t, \cdot)$—the solution of Eq. (2.4): for a Borel subset A in R^2,

$$P_{\mathbf{Y}}(A) = P(\mathbf{Y}(t) \in A) = \int_A \omega(t, \mathbf{y})\, d\mathbf{y}, \qquad t \geqslant 0. \tag{2.14}$$

Let $E_{\mathbf{Y}}$ denote the corresponding expectation, then (2.14) states simply that the velocity field $\mathbf{u}(t, \cdot)$ is the expectation of $\mathbf{K}(\cdot - \mathbf{Y}(t))$. We have explicitly that

$$\mathbf{u}(t, \mathbf{x}) = E_{\mathbf{Y}}[\mathbf{K}(\mathbf{x} - \mathbf{Y}(t))]. \tag{2.15}$$

Thus the velocity field $\mathbf{u}(t, \cdot)$ is completely determined by the distribution of $\mathbf{Y}(t)$. This connection enables us to solve the Navier–Stokes equations (2.1)–(2.2) by solving the SDE (2.4). Equations (2.4) and (2.15) constitute the basis for our study of random vortex methods.

Cutoff Functions. To approximate the velocity field, great care must be taken to avoid the singularity of the kernel $\mathbf{K} = \mathbf{K}(\mathbf{x})$ at the origin. For this purpose, we need to introduce a class of so-called cutoff functions. Let $\Phi = \Phi(\mathbf{x})$ be rapidly decreasing C^∞ function with integral one: $\int_{R^2} \Phi(\mathbf{x})\, d\mathbf{x} = 1$, and put

$$\Phi_\delta(\mathbf{x}) = \delta^{-2}\, \Phi(\mathbf{x}/\delta).$$

Then for every function $\mathbf{h} \in L^p(R^2)$ $(1 \leqslant p \leqslant \infty)$, we have $\mathbf{h} * \Phi \in C^\infty$, and

$$\mathbf{h}_\delta(\mathbf{x}) \equiv (\mathbf{h} * \Phi_\delta)(\mathbf{x}) \to \mathbf{h}(\mathbf{x}), \qquad \text{as} \quad \delta \to 0, \text{ in } L^p(R^2)$$

(see [19]). We call $\mathbf{u}_\delta \equiv \mathbf{u} * \Phi_\delta$ the regularization (or mollification) of $\mathbf{u}$, δ the cutoff for the function Φ_δ, and the family $\{\Phi_\delta\}$ the set of cutoff functions.

We apply this mollification to the velocity kernel $\mathbf{K}$, beginning with the stream function ψ. Since $\omega(t, \cdot)$ is assumed to decay rapidly at infinity, the stream function $\psi(t, \cdot) = (G * \omega)(t, \cdot)$ is in $L^1 \cap L^\infty$. Suppose that $G_\delta(\mathbf{x}) = (G * \Phi_\delta)(\mathbf{x})$ exists for every sufficiently small δ, then

$$(G_\delta * \omega)(t, \mathbf{x}) = (G * \Phi_\delta) * \omega = (G * \omega) * \Phi_\delta \to \psi(t, \mathbf{x}), \tag{2.16}$$

in the $L^1 \cap L^\infty$ sense as δ tends to 0. Furthermore, we have

$$\Delta G_\delta(\mathbf{x}) = \Phi_\delta(\mathbf{x}), \tag{2.17}$$

since $G = G(\mathbf{x})$ (see (2.8)) is the fundamental solution of the Laplace operator; and therefore we have according to (2.6)

$$\omega_\delta(t, \mathbf{x}) = \Delta(G_\delta * \omega) = (\Phi_\delta * \omega)(t, \mathbf{x}). \tag{2.18}$$

Now, as in (2.9) we define $\mathbf{K}_\delta = (K_{\delta 1}, K_{\delta 2})$ by

$$K_{\delta 1} = -\partial_2 G_\delta, \qquad K_{\delta 2} = \partial_1 G_\delta.$$

Then an approximation of the velocity field is given by

$$u_{\delta 1}(t, \mathbf{x}) = -\partial_2(G_\delta * \omega) = (K_{\delta 1} * \omega)(t, \mathbf{x}),$$
$$u_{\delta 2}(t, \mathbf{x}) = \partial_1(G_\delta * \omega) = (K_{\delta 2} * \omega)(t, \mathbf{x}).$$

Therefore, we have obtained a complete set of cutoff equations. In summary, let $\mathbf{u}_\delta$ denote $(u_{\delta 1}, u_{\delta 2})$; then we are solving Eqs. (2.4), (2.13), and (2.14) with $\mathbf{u}$ replaced by $\mathbf{u}_\delta$ and $\mathbf{K}$ replaced by $\mathbf{K}_\delta$. That is, we are considering

$$d\mathbf{Y} = \mathbf{u}_\delta(t, \mathbf{Y}(t))\, dt + \sqrt{2\nu}\, d\mathbf{W}_t, \qquad t \geqslant 0, \tag{2.19}$$

while the probability density ω of $\mathbf{Y}(t)$ satisfies

$$\omega_t(t, \mathbf{x}) + (\mathbf{u}_\delta \cdot \nabla)\, \omega(t, \mathbf{x}) = \nu \Delta \omega(t, \mathbf{x}) \tag{2.20}$$

and

$$\mathbf{u}_\delta(t, \boldsymbol{\sigma}) = \int \mathbf{K}_\delta(\mathbf{x}(t, \boldsymbol{\sigma}) - \mathbf{y}(t, \boldsymbol{\sigma}'))\, dP_{\mathbf{Y}}(\boldsymbol{\sigma}') \tag{2.21}$$

or, in analogy with (2.15),

$$\mathbf{u}_\delta(t, \mathbf{x}) = E_{\mathbf{Y}}[\mathbf{K}_\delta(\mathbf{x} - \mathbf{Y}(t))] \tag{2.22}$$

with (2.14) unchanged. Note that the cutoff velocity field $\mathbf{u}_\delta(t, \mathbf{x})$ satisfies the incompressibility condition (2.2) automatically by its definition; and (2.21) can be derived in exactly the same way as for (2.12). Similarly, (2.18) yields a probabilistic expression for the vorticity field;

$$\omega_\delta(t, \boldsymbol{\sigma}) = \int \Phi_\delta(\mathbf{x}(t, \boldsymbol{\sigma}) - \mathbf{y}(t, \boldsymbol{\sigma}'))\, dP_{\mathbf{Y}}(\boldsymbol{\sigma}'), \tag{2.23}$$

or

$$\omega_\delta(t, \mathbf{x}) = E_{\mathbf{Y}}[\Phi_\delta(\mathbf{x} - \mathbf{Y}(t))]. \tag{2.24}$$

Next we will derive random vortex methods based on Eqs. (2.19) and (2.22). Above all, we need an explicit kernel $\mathbf{K}_\delta$. If we choose Φ to be radially symmetric, it

follows from (2.17) that G_δ is also radially symmetric. Then, remembering the definition of $\mathbf{K}_\delta$, we have, by integrating (2.17),

$$\mathbf{K}_\delta(\mathbf{x}) = (-\partial_2, \partial_1)\, G_\delta(\mathbf{x}) = f_\delta(|\mathbf{x}|)\, \mathbf{K}(\mathbf{x}). \tag{2.25}$$

The similar idea was used by Beale and Majda [7] to design high order accuracy cutoff functions for the vortex method. In Table I, we list several commonly used function pairs f_δ, Φ derived by Beale and Majda, together with some other cutoff

TABLE I

Commonly Used Cutoff Functions

Chorin [13]:

$$f_\delta(r) = 1, \quad \text{if } r > \delta,$$
$$= r/\delta, \quad \text{if } r \leqslant \delta;$$

Milinazzo and Saffman [31]:

$$f_\delta(r) = 1, \quad \text{if } r > \delta,$$
$$= r^2/\delta^2, \quad \text{if } r \leqslant \delta;$$

Hald [24]:

$$\Phi(r) = \frac{1}{2\pi}\,[140(1-r)^3 - 420(1-r)^4 + 252(1-r)^5],$$

$$f_\delta(r) = \frac{r^2}{\delta^2}\left[14 - 105\frac{r^2}{\delta^2} + 196\frac{r^3}{\delta^3} - 140\frac{r^4}{\delta^4} + 36\frac{r^5}{\delta^5}\right] \text{ if } r \leqslant \delta, \quad = 1 \text{ if } r > \delta;$$

$$\Phi(r) = \frac{1}{2\pi}\left[280(1-r)^4 - 780(1-r)^5 + \frac{1400}{3}(1-r)^6\right],$$

$$f_\delta(r) = \frac{r^2}{3\delta^2}\left[56 - 630\frac{r^2}{\delta^2} + 1568\frac{r^3}{\delta^3} - 1680\frac{r^4}{\delta^4} + 864\frac{r^5}{\delta^5} - 175\frac{r^6}{\delta^6}\right] \text{ if } r \leqslant \delta, \quad = 1 \text{ if } r > \delta;$$

Kuhawara and Takami [27]:

$$\Phi(r) = \frac{1}{\pi}e^{-r^2}, \qquad f_\delta(r) = 1 - e^{-r^2/\delta^2};$$

Beale and Majda [7]:

$$\Phi(r) = \frac{1}{\pi}\left(2e^{-r^2} - \frac{1}{2}e^{-r^2/2}\right), \qquad f_\delta(r) = 1 - 2e^{-r^2/\delta^2} + e^{-r^2/2\delta^2};$$

$$\Phi(r) = \frac{1}{\pi}(2 - r^2)\,e^{-r^2}, \qquad f_\delta(r) = 1 - \left(\frac{r^2}{\delta^2} - 1\right)e^{-r^2/\delta^2}$$

functions, including those used by Chorin [13], Hald [24], Kuwahara and Takami [27], and Milinazzo and Saffman [31].

Discretized Methods. We can now derive the random vortex methods which are used in actual computation. Recall that the equations we are solving are (2.19) and (2.21). However, our approximation will be no longer restricted to the case where ω_0 is a probability density. First we write in a product form

$$\omega_0(\mathbf{x}) = \kappa(\mathbf{x})\, \pi_0(\mathbf{x})$$

so that $\pi(0, x) = \pi_0(x)$ is a probability density. Then we sample N independent random variables $\mathbf{Y}_j^{(0)}$, each with weight $\kappa_j = \kappa(\mathbf{Y}_j^{(0)})$, $1 \leqslant j \leqslant N$, according to the initial distribution $\pi_0(\cdot) = \pi(0, \cdot)$. Each pair $(\mathbf{Y}_j, \kappa_j)$ moving with fixed cutoffs (δ, Φ_δ) is called a vortex blob, or simply vortex.

Next we define an approximation to the $\mathbf{Y}^{(n)}$-velocity field,

$$\tilde{\mathbf{u}}_{\mathbf{Y}}^{(n)}(\mathbf{x}) = \frac{1}{N} \sum_{j=1}^{N} \mathbf{K}_\delta(\mathbf{x} - \mathbf{Y}_j^{(n)})\, \kappa_j; \tag{2.26}$$

this can be regarded as a Monte Carlo estimate for the integral (2.21). Other estimates are possible, but the one in (2.26) is the simplest. Although the samples $\{\mathbf{Y}_j^{(n)}\}$ are not independent due to their mutual interaction, the law of large numbers justifies the estimate because they are identically distributed [8]. From (2.23) or (2.24), an approximation for the vorticity is given by

$$\tilde{\omega}_{\mathbf{Y}}^{(n)} = \frac{1}{N} \sum_{j=1}^{N} \Phi_\delta(\mathbf{x} - \mathbf{Y}_j^{(n)})\, \kappa_j. \tag{2.27}$$

As we shall see, the expressions (2.26), (2.27) are approximations to the integrals (2.22) and (2.24) following the vortex blobs, provided that uniform grids are used initially. Following the notations used in the Appendix, we have immediately

Euler's Method. $\mathbf{Y}_j^{(n+1)} = \mathbf{Y}_j^{(n)} + \sqrt{2\Delta t\, \nu}\, \Gamma_j^{(n)} + \Delta t\, \tilde{\mathbf{u}}_{\mathbf{Y}}^{(n)}(\mathbf{Y}_j^{(n)})$. Because of its simplicity, this scheme has been used by Chorin in the study of slightly viscous flows [13] and of boundary layer approximations [14]. It is also one of the simplest schemes of fractional step type and of Runge–Kutta type. For more complicated schemes, we define the approximation to the $\mathbf{P}^{(n)}$-velocity field,

$$\tilde{\mathbf{u}}_{\mathbf{P}}^{(n)}(\mathbf{x}) = \frac{1}{N} \sum_{j=1}^{N} \mathbf{K}_\delta(\mathbf{x} - \mathbf{P}_j^{(n)})\, \kappa_j, \tag{2.28}$$

and the approximation to the $\mathbf{Q}^{(n)}$-velocity field,

$$\tilde{\mathbf{u}}_{\mathbf{Q}}^{(n)}(\mathbf{x}) = \frac{1}{N} \sum_{j=1}^{N} \mathbf{K}_\delta(\mathbf{x} - \mathbf{Q}_j^{(n)})\, \kappa_j. \tag{2.29}$$

The latter approximation will serve as an intermediate interaction between the velocity field and the random field $\mathbf{W}_t$ with an increment of the Wiener process entering $\mathbf{Q}^{(n)}$. Looking at Methods A_0 and B_0 in the Appendix, we define the following analogous schemes for Eqs. (2.19) and (2.21):

Method A.

$$\mathbf{P}_j^{(n)} = \mathbf{Y}_j^{(n)} + \tfrac{1}{2}\Delta t\, \tilde{\mathbf{u}}_{\mathbf{Y}}^{(n)}(\mathbf{Y}_j^{(n)}),$$
$$\mathbf{Q}_j^{(n)} = \mathbf{Y}_j^{(n)} + \tfrac{1}{2}\Delta t\, \tilde{\mathbf{u}}_{\mathbf{Y}}^{(n)}(\mathbf{Y}_j^{(n)}) + \tfrac{1}{2}\sqrt{2\Delta t\nu}\, \boldsymbol{\Gamma}_j^{(n)},$$
$$\mathbf{Y}_j^{(n+1)} = \mathbf{Y}_j^{(n)} + \sqrt{2\Delta t\nu}\, \boldsymbol{\Gamma}_j^{(n)} + \tfrac{1}{2}\Delta t\,[\tilde{\mathbf{u}}_{\mathbf{P}}^{(n)}(\mathbf{P}_j^{(n)}) + \tilde{\mathbf{u}}_{\mathbf{Q}}^{(n)}(\mathbf{Q}_j^{(n)})];$$

Method B.

$$\mathbf{P}_j^{(n)} = \mathbf{Y}_j^{(n)} + \tfrac{1}{2}\Delta t\tilde{\mathbf{u}}_{\mathbf{Y}}^{(n)}(\mathbf{Y}_j^{(n)}),$$
$$\mathbf{Q}_j^{(n)} = \mathbf{Y}_j^{(n)} + \tfrac{1}{2}\Delta t\tilde{\mathbf{u}}_{\mathbf{Y}}^{(n)}(\mathbf{Y}_j^{(n)}) + \tfrac{3}{2}\sqrt{2\Delta t\nu}\, \boldsymbol{\Lambda}_j^{(n)},$$
$$\mathbf{Y}_j^{(n+1)} = \mathbf{Y}_j^{(n)} + \sqrt{2\Delta t\nu}\, \boldsymbol{\Gamma}_j^{(n)} + \tfrac{1}{3}\Delta t[\tilde{\mathbf{u}}_{\mathbf{P}}^{(n)}(\mathbf{P}_j^{(n)}) + 2\tilde{\mathbf{u}}_{\mathbf{Q}}^{(n)}(\mathbf{Q}_j^{(n)})].$$

Note that μ in Methods A_0 and B_0 corresponds to $\sqrt{2\nu}$ in the above schemes since we are solving Eq. (2.19). The velocity field $\mathbf{u}_\delta(t_n, \cdot)$, given by (2.21), is determined from the flow at time t_n; therefore the time dependence is raised to the superscript in the approximation of the velocity fields. Note also that as ν tends to 0, these schemes become the vortex method based on the midpoint rule,

$$\mathbf{R}_j^{(n)} = \mathbf{X}_j^{(n)} + \tfrac{1}{2}\Delta t\tilde{\mathbf{u}}_{\mathbf{X}}^{(n)}(\mathbf{X}_j^{(n)}),$$
$$\mathbf{X}_j^{(n+1)} = \mathbf{X}_j^{(n)} + \Delta t\tilde{\mathbf{u}}_{\mathbf{R}}^{(n)}(\mathbf{R}_j^{(n)}),$$

which has been discussed by Anderson and Greengard [2] and proved to have a second-order accuracy by these authors. Convergence for the random vortex method has recently been studied by Goodman [21] and Marchioro and Pulvirenti [30], mainly for Euler's method. Also of interest are Hald [25] who proved, at an earlier time, convergence of a simplified model problem with vorticity creation and Roberts [34] who proved independently the convergence of a random vortex method for Burgers' equation.

3. The Model Problem and Variance Reduction

Consider the vorticity equation

$$\omega_t + (\mathbf{u}\cdot\nabla)\,\omega = \nu\Delta\omega, \tag{3.1}$$

subject to the initial condition

$$\omega_0(\mathbf{x}) = \Omega_0,\ |x| \leqslant a; \qquad = 0, \text{ elsewhere;} \tag{3.2}$$

where $\Omega_0 = 1/(\pi a^2)$ is so that $\omega_0(\mathbf{x})$, and therefore $\omega(t, \mathbf{x})$, is a probability density. Due to the symmetry of the initial $\omega(0, \mathbf{x})$, the convective term vanishes [4]; and therefore Eq. (3.1) reduces to the diffusion equation:

$$\omega_t(t, \mathbf{x}) = \nu \Delta \omega(t, \mathbf{x}). \tag{3.3}$$

The initial value problem (3.3) and (3.2) allows for the explicit solution

$$\omega(t, \mathbf{x}) = \int_{R^2} D_t(\mathbf{x} - \mathbf{y})\, \omega_0(\mathbf{y})\, d\mathbf{y}, \tag{3.4}$$

where $D_t(\mathbf{x})$ is the diffusion kernel. For later use, we compute the initial $\mathbf{u}_0$ of the flow. By (2.6), the radial symmetry of the initial vorticity field implies that $\psi_0(\mathbf{x}) = \psi_0(r)$, $r = |x|$, and in polar coordinates,

$$\frac{1}{r}\frac{d}{dr}\left(r^2 \frac{d\psi_0}{dr}\right) = \Omega_0, \qquad |\mathbf{x}| \leqslant a; \qquad = 0, \qquad \text{elsewhere.}$$

Direct integration of this equation twice, followed by the use of the formulas in (2.5), gives

$$\mathbf{u}_0(\mathbf{x}) = \begin{cases} \frac{1}{2}\Omega_0(-x_2, x_1), & \text{if } |\mathbf{x}| \leqslant a, \\ \frac{1}{2}\dfrac{a^2}{|\mathbf{x}|^2}\Omega_0(-x_2, x_1), & \text{if } |\mathbf{x}| > a. \end{cases} \tag{3.5}$$

Now we consider the problem of investigating the accuracy of the methods derived in the previous section. It seems that the simplest and least expensive way to compare the accuracy of stochastic numerical schemes is to use them to estimate functionals which can be evaluated explicitly. This indicates the accuracy both of the computed velocity field and of the vortex distribution. Following [31, 33], we consider integrals of the form:

$$E_{\mathbf{Y}}[g(\mathbf{Y}(t))] = \int_{R^2} g(\mathbf{y})\, \omega(t, \mathbf{y})\, d\mathbf{y}, \tag{3.6}$$

where $\mathbf{Y}(t)$ is the solution of Eq. (2.4) and $E_{\mathbf{Y}}$ denotes the corresponding expectation. Integral (3.6), like the velocity field is a functional of the Wiener process $\mathbf{W}_t$. We take first $g(\mathbf{x}) = |\mathbf{x}|^2$. Substituting (3.4) in (3.6) and performing the integration yields

$$U(t) = \tfrac{1}{2}a^2 + 4\nu t. \tag{3.7}$$

Then put $g(\mathbf{x}) = \exp(-|\mathbf{x}|^2)$, and we have

$$V(t) = \frac{1}{a^2}\exp\left(-\frac{a^2}{4\nu t + 1}\right). \tag{3.8}$$

Since we are numerically solving an SDE (i.e., (2.19)), all the information available at the time t_n is the set of samples $\mathbf{Y}_j^{(n)}$. Therefore, we would Monte Carlo estimate the functionals considered. The simplest estimate for (3.7) or (3.8) is the usual estimate

$$\frac{1}{N}\sum_{j=1}^{N} g(\mathbf{Y}_j^{(n)}). \tag{3.9}$$

An estimate of this kind can be justified, as mentioned before, by the law of large numbers, despite the lack of independence among the samples $\mathbf{Y}_j^{(n)}$ due to their mutual interaction through the velocity field.

Variance Reduction. A Monte Carlo estimate creates statistical error—called error variance due to imperfect sampling. Since the error variance for the estimate in (3.9) is usually proportional to $1/N$ (see [26]), a substantial error of order $1/\sqrt{N}$ is due to the Monte Carlo calculation. It is then clear that, in order to exhibit the accuracy of a numerical scheme, the error variance produced by Monte Carlo estimation should be made as small as possible. The method is to utilize the non-anticipating property of the solutions of SDEs. In view of (3.6), we form the sum

$$g(\mathbf{Y}(t_n)) = g(\mathbf{Y}(0)) + \sum_{k=1}^{n-1} [g(\mathbf{Y}(t_{k+1})) - g(\mathbf{Y}(t_k))].$$

Its numerical analogy is then given by

$$g^{(n)} = g^{(0)} + \sum_{k=1}^{n-1} (g^{(k+1)} - g^{(k)}), \tag{3.10}$$

where $g^{(k)} \equiv g(\mathbf{Y}^{(k)})$, $\mathbf{Y}^{(k)}$ is the numerical solution of the SDE (2.19). We further observe that all the schemes considered (Euler, Methods A and B) are of the form:

$$\mathbf{Y}^{(k+1)} = \mathbf{Y}^{(k)} + \sqrt{2\Delta t \nu}\,\mathbf{\Gamma}^{(k)} + \Delta t\,\mathbf{\Psi}^{(k)}.$$

The non-anticipating property implies that the numerical solution $\mathbf{Y}^{(k)}$ is independent of the newly input increment $\mathbf{\Gamma}^{(k)}$. This suggests the following Taylor expansion for each summand in (3.10)

$$g_j^{(k+1)} = g_j^{(k)} + g_{,l}^{(k)}|_j [\sqrt{2\Delta t \nu}\,\Gamma^{(k),l} + \Delta t\,\Psi^{(k),l}] + \Delta t \nu g_{,lp}^{(k)}|_j \Gamma^{(k),l}\Gamma^{(k),p} + \quad . \tag{3.11}$$

Here the summation convention is adopted, and the subscript j means that the sample $\mathbf{Y}_j$ is used, and subscripts with a comma denote differentiation. Hence we need to assume smoothness on g. For convenience, we omit the index j temporarily. Move the two terms on the right-hand side of (3.11) to the left and denote by $G^{(k+1)}$ the resultant expression, then

$$\begin{aligned} G^{(k)} &= g^{(k+1)} - g^{(k)} - \sqrt{2\nu\,\Delta t}\, g_{,l}^{(k)}\,\Gamma^{(k),l}, \\ &= \Delta t\, g_{,l}^{(k)}\,\Psi^{(k),l} + \Delta t \nu g_{,lp}^{(k)}\,\Gamma^{(k),l}\Gamma^{(k),p} + \quad . \end{aligned} \tag{3.12}$$

Note the independence between $g^{(k)}$ and $\Gamma^{(k),l}$, the latter being of standard Gaussian distribution. Taking expectations on both sides of the first equality and summing the results over k from 0 on $n-1$, we have immediately

$$E[g(\mathbf{Y}^{(n)})] = E[g(\mathbf{Y}^{(0)})] + \sum_{k=1}^{n-1} E[G^{(k)}],$$

which is equivalent to

$$E[g(\mathbf{Y}^{(k+1)})] = E[g(\mathbf{Y}^{(k)})] + E[G^{(k)}]. \tag{3.13}$$

This is a recursive relation between $E[g(\mathbf{Y}^{(k)})]$ and $E[g(\mathbf{Y}^{(k+1)})]$. In this way the functionals to be computed are successively linked, and an estimate for each $E[G^{(k)}]$ can be designed, according to the first identity of (3.12), as

$$\frac{1}{N}\sum_{j=1}^{N} [G_j^{(k)}] \equiv \frac{1}{N}\sum_{j=1}^{N} [g_j^{(k+1)} - g_j^{(k)} - \sqrt{2\Delta t \nu}\, g_{j,l}^{(k)} \Gamma^{(k),l}], \tag{3.14}$$

where, again, $g_j^{(k)}$ means that the sampling solution $\{\mathbf{Y}_j^{(k)}\}$ is used. The reason for using this form is then clear: the usual estimate for $g(\mathbf{Y}^{(n)})$ is equivalent to summing up estimates of the same kind for $g(\mathbf{Y}^{(k+1)}) - g(\mathbf{Y}^{(k)})$ over k from 0 to $n-1$, while the estimate for $g(\mathbf{Y}^{(n)})$ constructed through $G^{(k)}$ eliminates the dominating sampling error due to the existence of the term $\sqrt{2\Delta t \nu}\, g_{,l}^{(k)} \Gamma^{(k),l}$. More complicated estimates can be constructed through the use of Hermite polynomials [12, 29].

4. Numerial Implementation

In this section we present the numerical results of performing Monte Carlo calculation of the two functionals considered in Section 3 (cf. (3.6)–(3.8)). The Monte Carlo estimates employed are the usual one in (3.9) and the modified one in (3.13), (3.14). We consider the three methods presented in Section 2: Euler's method and Methods A and B. In studying the accuracy of these schemes, we wish to concern ourselves only with the time discretization error but we must consider the random sampling error as well. Therefore one aim of this section is to weigh the relative significance of these errors and estimate their interaction.

Before any meanful discussion, we have to specify the following parameters and conditions:

(i) the time step size (Δt),

(ii) the discretization of the initial vorticity field,

(iii) the cutoff δ and the cutoff function f_δ, and

(iv) the sampling algorithm for the variables: $\{\Gamma_j^{(n)}\}$ and $\{\Lambda_j^{(n)}\}$.

To exhibit the accuracy of the numerical schemes, we use three time step sizes $\Delta t = 0.2$, 0.1, and 0.05 for Monte Carlo computation of each functional under consideration. As we have noted in Section 3, the time step size may also affect the Monte Carlo computation, and this point will be discussed further later.

The number of vortex blobs used is actually determined by the spatial discretization of the support—a disk of radius a—of the initial vorticity field. Since we are mainly concerned with the two errors mentioned above, the initial error due to discretization will be minimized as follows. The radius a of the disk is chosen to be 0.5, and therefore the initial vorticity is given by

$$\omega_0(\mathbf{x}) = \frac{4}{\pi}, \qquad |\mathbf{x}| \leqslant a; \qquad = 0, \qquad \text{elsewhere},$$

so that the total vorticity is 1. Following Roberts [33], we will uniformly distribute the initial vortex blobs; thus we set $\omega_0(\mathbf{x}) = \kappa(\mathbf{x})\, \pi_0(\mathbf{x})$, where $\pi_0 = \pi_0(\mathbf{x})$ is uniform over the unit square centered at the origin. The corresponding weight function $\kappa = \kappa(\mathbf{x})$ is therefore identical to the initial vorticity field: $\kappa(\mathbf{x}) = \omega_0(\mathbf{x})$. Next we partition the unit square into a lattice of sidelength $\frac{1}{32} = 0.03125$, which is comparable to that used by Perlman [32] in a vortex method. The center of each square L_j is the position of a vortex blob which carries the average weight of the square;

$$\kappa_j = \int_{L_j} \kappa(\mathbf{y})\, d\mathbf{y}/\text{area } (L_j).$$

The weights $\{\kappa_j\}$ used here are somewhat different from those defined before; the previous ones were obtained by evaluating the weight function $\kappa = \kappa(\mathbf{x})$ at the initial positions of the vortex blobs. Consequently the total weight is exact;

$$\sum_j \kappa_j \cdot \text{area}(L_j) = \sum_j \int_{L_j} \kappa(\mathbf{y})\, d\mathbf{y} = \int \kappa(\mathbf{y})\, d\mathbf{y}.$$

It should be noted, however, that only those vortex blobs with nonzero average weights κ_j enter the actual computation, because the others contribute nothing to the computed velocity fields. This explains why the number N of vortex blobs that we actually use is 856 instead of the number 1024 of squares. Therefore, it suffices to determine the initial velocity field for those vortex blobs with nonzero weights, from (3.5) with $\Omega_0 = 4/\pi$,

$$\mathbf{u}_0(\mathbf{x}) = \frac{1}{2}\, \Omega_0(-x_2, x_1) = \frac{2}{\pi}\, (-x_2, x_1).$$

Furthermore, we specify the cutoff δ and the function f_δ. The cutoffs serve as measures of the size of vortex blobs and of the interactions among them. These factors play a major role in determining the accuracy of a vortex method (see, e.g.,

[13, 24]). In the present paper, we choose δ to be the sidelength of an initial square and f_δ to be the fourth-order kernel of Beale and Majda ([7]). We use a fourth-order kernel to minimize the cutoff-induced error.

To simulate the Gaussian random variables $\Gamma_j^{(n)}$ and $\Lambda_j^{(n)}$, we recall that their correlation matrix is $N(0, 1/2I_2)$, and write

$$\Gamma_j^{(n)} = \mathbf{N}_{1j}^{(n)}, \qquad \Lambda_j^{(n)} = \frac{1}{2}\mathbf{N}_{1j}^{(n)} + \frac{\sqrt{3}}{6}\mathbf{N}_{2j}^{(n)},$$

where $\mathbf{N}_{1j}^{(n)}$ and $\mathbf{N}_{2j}^{(n)}$ are two independent Gaussian random variables. A two-component Gaussian variable $\mathbf{N} = (N_1, N_2)$ may be sampled according to the following formulas [26]:

$$N_1 = \cos(2\pi U_1)[-2\log(U_2)]^{1/2},$$
$$N_2 = \sin(2\pi U_1)[-2\log(U_2)]^{1/2},$$

TABLE II

Example 1, $N = 856$. Numerical Results of Computing $\int |\mathbf{y}|^2 \omega(t, \mathbf{y})\, d\mathbf{y}$: for Each Scheme, Errors for the Usual and the Modified Monte Carlo Estimates Are Listed in Order in Exponential Form

Δt	Euler's		Method A		Method B	
$t = 1.0$, Exact value = 1.330-1						
0.2000	8.267-3	9.430-3	−1.424-3	−2.684-4	−1.475-4	2.196-5
0.1000	4.789-3	3.580-3	−6.035-4	−2.405-4	−1.573-3	−9.270-5
0.0500	2.574-3	2.311-3	1.223-4	−1.409-4	−1.326-3	−6.156-5
$t = 2.0$, Exact value = 1.410-1						
0.2000	1.689-2	1.764-2	1.459-3	−5.797-4	−1.227-3	−6.603-5
0.1000	9.938-3	9.060-3	5.673-4	−3.057-4	−5.726-4	−6.798-5
0.0500	1 556-3	4.521-3	−3.135-3	−2.330-4	−4.231-3	−4.898-5
$t = 3.0$, Exact Value = 1.490-1						
0.2000	2.670-2	2.564-2	−3.256-4	−5.701-4	1.083-4	8.836-6
0.1000	1.290-2	1.316-2	−4.989-4	−3.660-4	−3.021-3	−1.395-4
0.0500	−1.840-3	6.698-3	−5.043-3	−2.573-4	−5.457-3	−7.560-6
$t = 4.0$, Exact value = 1 570-1						
0.2000	3.457-2	3.254-2	8.940-4	−8.191-4	2.073-3	−1.719-4
0.1000	1.506-2	1.712-2	−1.969-3	−4.138-4	−3.721-3	7.749-5
0.0500	6.231-3	8.872-3	−2.789-3	−1.924-4	−5.931-3	−5.898-5

where U_1 and U_2 are two independent (scalar) uniform random variables over the unit interval [0, 1]. The above sampling procedure is done for each vortex blob j at each time step (n) independently, Finally we choose the viscosity $\nu = 0.002$, within the range of a typical slightly viscous flow.

Now we analyze the numerical results. For each functional, we list the computed results in four tables at the times $t = 1$, 2, 3, and 4 (see Tables II and III). The total time interval $(T = 4)$ is comparable to the period of the corresponding nonviscous flow [32]. For each scheme, there are two subcolumns, containing results obtained by the usual estimate and the modified estimate, respectively. We note that for Euler's method, the errors produced by these two estimates are about of the same order, and particularly the modified one exhibits clearly the first-order accuracy of this method.

The situation for Methods A and B is quite different from that for Euler's method. First we observe that the errors are drastically reduced if the modified estimate is used. Let us focus on the ratios of the errors for the modified estimate to the errors for the usual estimate. Then we find that the absolute values of the ratios for Method A are about from 0.1 to 1, while for Method B the values are about

TABLE III

Example 2, $N = 856$. Numerical Results of Computing $\int e^{-|y|^2} \omega(t, y)\, dy$: for Each Scheme, Errors for the Usual and the Modified Monte Carlo Estimates Are Listed in Order in Exponential Form

Δt	Euler's		Method A		Method B	
$t = 1.0$, Exact value = 8.7861-1						
0.2000	−5.382-3	−7.909-3	2.722-3	2.008-4	1.564-3	−4.661-5
0.1000	−2.496-3	−3.842-3	1.573-3	2.035-4	2 752-3	4.542-5
0.0500	−7.046-4	−1.960-3	1.355-3	1.006-4	2.526-3	9.954-6
$t = 2.0$, Exact value = 8.7251-1						
0.2000	−1.246-2	−1.456-2	2.632-3	4.309-4	−2.500-3	−4.172-6
0.1000	−6.807-3	−7.533-3	9.334-4	2.111-4	1.814-3	−2.331-5
0.0500	1.023-4	−3.774-3	4.010-3	1.722-4	4.760-3	−3.219-5
$t = 3.0$, Exact value = 8.6649-1						
0.2000	−2.025-2	−2.081-2	1.047-3	3.826-4	1.235-3	−9.602-5
0.1000	−8.937-3	−1.075-2	1.983-3	2 765-4	3.720-3	1.597-5
0.0500	−4.585-4	−5.542-3	5.223-3	1.639-4	−5.645-3	−1.494-4
$t = 4.0$, Exact value = 8.6055-1						
0.2000	−2.620-2	−2.600-2	5.181-4	5.488-4	−3.717-4	9.692-5
0.1000	−1.078-2	−1.382-2	2.967-3	2.881-4	4.055-3	−2.121-4
0.0500	−4.171-3	−7.295-3	3.167-3	4.804-5	6.000-3	−1.868-4

from 0.01 to 0.3. However, we also observe that there is no clear dependence of the reduced errors on the time step sizes; this indicates the existence of strong interactions between the time discretization errors and the random sampling errors. As we have noted, the fact that the time step size enters into both the local truncation errors and the Monte Carlo modified estimate complicates the mutual interactions.

Nevertheless, in any case, the numerical results reveal that both Method A and Method B are more accurate than Euler's method by one power of the time step size. The seeming superiority of Method B can be understood as follows. Since the accuracy of Method B_0 has been proved to be the order 1.5 in the L_2 sense, it is very likely that Method B would exhibit a second-order accuracy in the weak sense due to the non-anticipating property of SDEs. On the other hand, Method A_0 accumulates local truncation errors in an invisible way and possibly does not produce these errors with small uniform bounds, though the order of the local truncation error at each time step is three in the weak sense. We used the word "invisible" because of the lack of (accurate) estimation of the accumulated error. We do not observe, in the present analysis, the dependence of errors on the viscosity. However, we expect that for small viscosity, the random effects are small as well, and our schemes would produce even better results. Finally, we would like to mention on this connection that the viscosity ($\nu = 0.002$) used in this study is slightly larger than those used elsewhere (e.g., [31, 33]).

5. Concluding Remarks

The vorticity in two-dimensional incompressible flow evolves under a convection term and a diffusion term, exactly corresponding to the drift vector and Brownian part of a diffusion process. Therefore, it is reasonable that one can derive accurate numerical methods for solving the Navier–Stokes equations from those known for SDEs. This has been verified in the present article (Methods A and B). However, as we have noted, further study in this direction should include both the accuracy of numerical schemes and variance reduction techniques.

Of equal interest is the generalization of the results to three dimensions, where an immediate hindrance is encountered: the emergence of vortex stretching. This important physics is not attached to a diffusion process, and therefore to an SDE, and should be treated separately. Moreover, since the vorticity in three dimensions is no longer scalar, the analogy with what we have done is not clear, there may be a stochastic calculus which can model the physics of vortex stretching. However, Esposito and Pulvirenti [18] have more or less extended the three-dimensional convergence results of Beale and Majda [5] to the stochastic case, using a splitting algorithm.

APPENDIX

SDE and Numerical Methods. The simplest d-dimensional SDE is given by

$$d\mathbf{Y} = \mathbf{f}(t, \mathbf{Y})\, dt + \mu\, d\mathbf{W}_t,$$

where $\mathbf{f} = \mathbf{f}(t, \mathbf{x})$ is called the drift vector and $\mu \geqslant 0$ the diffusion coefficient. $\mathbf{W}_t = \mathbf{W}(t, \cdot)$ is called a Wiener process or Brownian motion. A Wiener process is a Gaussian stochastic process with independent increments starting at the origin. More precisely, $\mathbf{W}_t$ satisfies

(i) $\mathbf{W}_0 = 0$,

(ii) $\mathbf{W}_t - \mathbf{W}_s$, $s \leqslant t$, is of Gaussian distribution $N(0, (t-s)\, I_d)$,

(iii) $\mathbf{W}_s$ is independent of $\mathbf{W}_t - \mathbf{W}_s$, $s \leqslant t$.

It can be shown that a Brownian path is nowhere differentiable. Therefore, a solution of the SDE should be interpreted in the sense of integration,

$$\mathbf{Y}(t) = \mathbf{Y}(0) + \int_0^t \mathbf{f}(s, \mathbf{Y}(s))\, ds + \mu \mathbf{W}_s.$$

From the property of the Wiener process, it is conceivable that the solution to an SDE is also independent of later increments of the Wiener process, which is called the nonanticipating property of the solution.

Numerical Methods. Due to the existence of a Wiener process, it is not clear how to extend the techniques for ODEs to SDEs. In [9, 10], we proposed two methods of Runge–Kutta type for numerical integration of SDEs. Let Δt denote the time step and set $t_n^* = t_n + \frac{1}{2}\Delta t$, we have

Method A_0:

$$\begin{aligned}
\mathbf{P}^{(n)} &= \mathbf{Y}^{(n)} + \tfrac{1}{2}\Delta t\, \mathbf{f}(t_n, \mathbf{Y}^{(n)}),\\
\mathbf{Q}^{(n)} &= \mathbf{Y}^{(n)} + \tfrac{1}{2}\Delta t\, \mathbf{f}(t_n, \mathbf{Y}^{(n)}) + \tfrac{1}{2}\sqrt{\Delta t}\, \mu \boldsymbol{\Gamma}^{(n)},\\
\mathbf{Y}^{(n+1)} &= \mathbf{Y}^{(n)} + \sqrt{\Delta t}\, \mu \boldsymbol{\Gamma}^{(n)} + \tfrac{1}{2}\Delta t\, [\mathbf{f}(t_n^*, \mathbf{P}^{(n)}) + \mathbf{f}(t_n^*, \mathbf{Q}^{(n)})];
\end{aligned}$$

Method B_0:

$$\begin{aligned}
\mathbf{P}^{(n)} &= \mathbf{Y}^{(n)} + \tfrac{1}{2}\Delta t\, \mathbf{f}(t_n, \mathbf{Y}^{(n)}),\\
\mathbf{Q}^{(n)} &= \mathbf{Y}^{(n)} + \tfrac{1}{2}\Delta t\, \mathbf{f}(t_n, \mathbf{Y}^{(n)}) + \tfrac{3}{2}\sqrt{\Delta t}\, \mu \boldsymbol{\Lambda}^{(n)},\\
\mathbf{Y}^{(n+1)} &= \mathbf{Y}^{(n)} + \sqrt{\Delta t}\, \mu \boldsymbol{\Gamma}^{(n)} + \tfrac{1}{3}\Delta t\, [\mathbf{f}(t_n^*, \mathbf{P}^{(n)}) + 2\mathbf{f}(t_n^*, \mathbf{Q}^{(n)})].
\end{aligned}$$

$\boldsymbol{\Gamma}^{(n)}$ is a Gaussian random variable of $N(0, I_d)$ and $\boldsymbol{\Lambda}^{(n)}$ of $N(0, \frac{1}{3} I_d)$. Method B_0 is proven to have the order 1.5 in the L_2 sense while Method A_0 has local truncation errors of order 3 in the weak sense. Note that as μ tends to 0, these schemes become the Runge–Kutta method based on the midpoint rule for ODEs.

ACKNOWLEDGMENTS

The author wishes to thank Professors A. Chorin, O. Hald, and T. Beale for helpful comments and discussions. This work was carried out while the author was a Postdoctoral Fellow at the Lawrence Berkeley Laboratory, University of California, Berkeley, with partial support from the Applied Mathematics subprogram of the Office of Energy Research, U.S. Department of Energy under Contract DE-AC3-76SF00098.

REFERENCES

1. C. ANDERSON, Ph.D. thesis, Math. Department, University of California, Berkeley, 1983 (unpublished).
2. C. ANDERSON AND C. GREENGARD, *SIAM J. Numer. Anal.* **22**, 413 (1985).
3. L. ARNOLD, *Stochastic Differential Equations* (Wiley-Interscience, New York, 1974).
4. G. K. BATCHELOR, *An Introduction to Fluid Dynamics* (Cambridge University Press, Cambridge, 1967).
5. J. T. BEALE AND A. MAJDA, *Math. Comput.* **39**, 1 (1982).
6. J T. BEALE AND A. MAJDA, *Math. Comput.* **39**, 29 (1982).
7. J. T. BEALE AND A. MAJDA, PAM Report 178, University of California, Berkeley, 1983 (unpublished).
8. P. BILLINGSLEY, *Probability and Measure* (Wiley-Interscience, New York, 1979).
9. C. C. CHANG, Ph.D. thesis, Math Department, University of California, Berkeley, 1985 (unpublished)
10. C. C CHANG, *Math. Comput.* **49**, 523 (1987).
11. P. CHERNOFF, *J. Funct. Anal.* **2**, 238 (1968).
12. A. J. CHORIN, *J. Comput. Phys.* **8**, 472 (1971)
13. A. J. CHORIN, *J. Fluid Mech.* **57**, 785 (1973)
14. A. J CHORIN, *J. Comput. Phys.* **27**, 428 (428).
15. A. J. CHORIN, *SIAM J. Sci. Statist. Comput.* **1**, 1 (1980).
16. A. J. CHORIN AND P. BERNARD, *J. Comput. Phys.* **13**, 423 (1973).
17. A J. CHORIN, T. J. R. HUGHES, M F. MCCRACKEN, AND J. E MARSDEN, *Commun Pure Appl. Math.* **31**, 205 (1978).
18. R. ESPOSITO AND M PULVIRENTI, preprint, 1985 (unpublished).
19. G. B. FOLLAND, *Introduction to Partial Differential Equations* (Princeton Univ. Press, Princeton, NJ, 1976.
20. A F. GHONIEM, A. J CHORIN, AND A. K. OPPENHEIM, *Philos. Trans. R. Soc. London A* **304**, 303 (1983)
21. J. GOODMAN, preprint, 1985 (unpublished).
22. C. GREENGARD, Ph.D. thesis, Math. Department, University of California, Berkeley, 1984 (unpublished).
23. O HALD AND V. M. DEL PRETE, *Math. Comput.* **32**, 791 (1978).
24. O. HALD, *SIAM J. Numer. Anal.* **16**, 716 (1979).
25. O. HALD, *SIAM J. Sci. Statist. Comput.* **7**, 1373 (1986).
26. J. M. HAMMERSLEY AND D. C HANDSCOMB, *Monte-Carlo Methods* (Methuen, London, 1985).
27. K. KUHAWARA AND H. TAKAMI, *J. Phys. Soc. Japan* **34**, 247 (1973).
28. A LEONARD, *Ann. Rev. Fluid Mech.* **17**, 523 (1985).
29. F. H. MALTZ AND D. L. HITZL, *L Comput. Phys.* **32**, 345 (1979).
30. C. MARCHIORO AND M. PULVIRENTI, Lecture Notes in Physics, Vol. 203 (Springer-Verlag, Berlin, 1984).
31. F. MILINAZZO AND P G. SAFFMAN, *J. Comput. Phys.* **23**, 380 (1977).

32 M B. PERLMAN, Ph.D. thesis, Math. Department, University of California, Berkeley, 1983 (unpublished).
33. S. G. ROBERTS, *J. Comput. Phys.* **58**, 29 (1985).
34. S. G. ROBERTS, Ph.D. thesis, Math. Department, University of California, Berkeley, 1985 (unpublished).
35. J. A. SETHIAN, *J. Comput. Phys.* **54**, 425 (1984).
36. G. STRANG, *SIAM J. Numer. Anal.* **5**, 506 (1968).